Written Exam for the M.Sc. in Economics Autumn 2012 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **15/1-2013**

3-hour open book exam.

Please note there are a total of 11 questions which should **all be replied to**. That is, 6 questions under *Question A*, and 5 under *Question B*.

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question A:

Question A.1: Consider the well-known Gaussian ARCH(1) model

$$x_t = \sigma_t z_t, \tag{A.1}$$

with $z_t \sim i.i.d. N(0,1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2, \quad \omega > 0, \quad \alpha \geq 0.$$

Suppose x_t is stationary with $E(x_t^2) < \infty$, and define $\gamma := E(x_t^2)$. Find an expression for γ in terms of ω and α, and show that σ_t^2 can be re-written as

$$\sigma_t^2 = \gamma(1-\alpha) + \alpha x_{t-1}^2. \tag{A.2}$$

Question A.2: Show that x_t is stationary and weakly mixing with $E(x_t^4) < \infty$ if $\alpha < 1/\sqrt{3} \approx 0.58$.

Question A.3: In the following we define the vector of parameters in the model as $\theta = (\gamma, \alpha)'$. State the log-likelihood contribution at time t, $l_t(\theta)$ in terms of (A.1) and (A.2). Show that the score in the direction of α evaluated at the true parameters $\theta_0 = (\alpha_0, \gamma_0)$ is (up to a constant) given by:

$$\left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} = \tfrac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)}\left(1 - z_t^2\right)$$

Question A.4: Show that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} \xrightarrow{D} N(0, \xi),$$

Explain what ξ is.

Question A.5: Instead of estimating $\theta = (\gamma, \alpha)'$ by MLE, one could apply so-called *variance targeting* estimation, which can be described in the following way: First, γ is estimated by the simple empirical moment estimator,

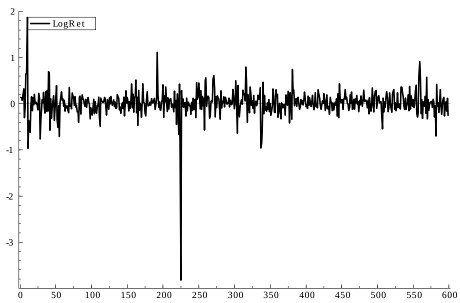
$$\hat{\gamma}_{VT} = \tfrac{1}{T} \sum_{t=1}^{T} x_t^2.$$

Next, α is estimated by QMLE with $\gamma = \widehat{\gamma}_{VT}$, that is $\widehat{\alpha}_{VT}$ maximizes $\sum_{t=1}^{T} l_t(\widehat{\gamma}_{VT}, \alpha)$.

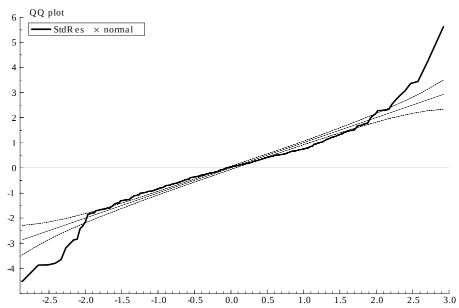
Show that $\hat{\gamma}_{VT}$ is consistent for γ_0 and state under which assumption(s) this holds.

Question A.6: Consider the graph below of log-returns series and output below in Table A.1 from estimation of the Gaussian ARCH(1) model by VTE together with a QQ-plot of standardized residuals.
Explain why you can conclude the model is misspecified.
Despite the fact that the model is misspecified it may be argued that the reported $\hat{\gamma}_{VT}$ can be used as a good estimate for the long-run variance. Why and under which conditions?
Suggest a possible alternative model which you believe would be better for these data.

Table A.1	
Parameter estimates VTE:	$\hat{\alpha}_{VT} = 0.61$ $\hat{\gamma}_{VT} = 0.04$
Standardized residuals: $\hat{z}_t = x_t/\hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	p-value: 0.00
LM ARCH test in $\hat{z}_t$:	p-value: 0.25



Daily log-returns $t = 1, 2, ..., T = 600$



QQ-plot: $\hat{z}_t$ solid fat black line.
Dotted lines 95%-conf. bands
for Gaussian distribution.

Question B:

Question B.1: Consider the return of the portfolio-series y_t in Figure B.1 with t=1,2,...,T=1000.

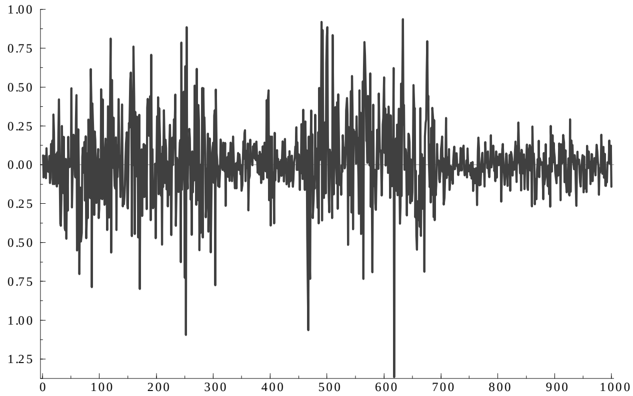


Figure B.1: Portfolio returns, y_t

Estimation with a 2-state switching volatility model, gave the following output in the usual notation in terms of the transition matrix $P = (p_{ij})_{i,j=1,2}$ and smoothed standardized residuals $\hat{z}_t^*$:

$\hat{P}$, QMLE of P:	$\hat{p}_{11} = 0.97$ $\quad \hat{p}_{22} = 0.99$
	p-values for LM tests based on $\hat{z}_t$:
LM-test for Normality:	0.06
LM-test for no ARCH:	0.10

What would you conclude on the basis of the output and the graph?

Question B.2: In order to compute Value-at-Risk (VaR) the following ARCH-type model was proposed:

$$y_t = \sigma_{s_t,t} z_t \tag{B.1}$$

Here z_t are iidN$(0,1)$ and indpendent of the unobserved iid variable s_t, $s_t \in \{1,2\}$, where $p = P(s_t = 1) = 1 - P(s_t = 2)$. Moreover,

$$\sigma_{1,t}^2 = \omega + \alpha y_{t-1}^2 \quad \text{and} \quad \sigma_{2,t}^2 = \gamma. \tag{B.2}$$

Thus the parameters of the model are $\theta = (\omega, \alpha, \gamma, p)$ with $\omega, \gamma > 0$, $\alpha \geq 0$ and $p \in (0,1)$.

Interpret the model and write the log-likelihood function $L_T(\theta)$ in terms of θ.

Question B.3: Differentiating $L_T(\theta)$ with respect to α gives,

$$\partial L_T/\partial\alpha = \sum_{t=1}^{T} p_t^* \left(1 - \tfrac{y_t^2}{\sigma_{1,t}^2}\right) \tfrac{y_{t-1}^2}{\sigma_{1,t}^2}, \qquad \text{where } p_t^* = \tfrac{p f_1(y_t|y_{t-1})}{p f_1(y_t|y_{t-1}) + (1-p) f_2(y_t|y_{t-1})}$$

Here $f_1(y_t|y_{t-1})$ is the Gaussian density corresponding to $\mathrm{N}(0, \sigma_{1,t}^2)$, and $f_2(y_t|y_{t-1})$ the Gaussian density corresponding to $\mathrm{N}(0, \sigma_{2,t}^2)$.

Discuss how to find the MLE of p and α (keeping the other parameters ω and γ fixed to ease the discussion).

Question B.4: The 5% conditional value at risk, cVaR_t is here defined as the cVaR_t which satisfies:

$$P\left(y_t \leq -\mathrm{cVaR}_t \mid y_{t-1}, ..., y_1\right) = 5\%.$$

Explain how you would use the proposed model in (B.1)-(B.2) to compute this if p is known and fixed with $p = 1$.

Discuss the difference between the cVaR_t here and the one computed using an iid Gaussian model for the returns y_t instead of the ARCH type here.

Question B.5: Explain how would you compute the cVaR_t is Question B.5 in the general case of (B.1)-(B.2) where p is also estimated together with ω, γ and α.